

# Explicit expressions for optical scalars in gravitational lensing from general matter sources

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We present explicit expressions for the optical scalars and the deflection angle in terms of the energy-momentum tensor components of matter distributions. Our work generalizes standard references in the literature where normally stringent assumptions are made on the sources.

## I. INTRODUCTION

Gravitational lensing has become a significant tool to make progress in our knowledge on the matter content of our Universe. In particular, there is a large number of works that use gravitational lensing techniques in order to know how much mass are in galaxies or clusters of galaxies. One of the most exiting results was to reaffirm the need for some kind of dark matter, that appears to interact with the barionic matter only through gravitation.

The question in which there is yet not general agreement is on the very nature of this dark matter. The most common conception is that it is based on collisionless particles[Wei08], and where the pressures are negligible. However in the context of cosmological studies, one often recurs to models of dark matter in terms of scalar fields. There is also the possibility that dark matter were described in terms of spinor fields.

One method to study the nature of dark matter consists in observing the deformation of images of galaxies behind a matter distribution that is the source of a gravitational lens.

In many astrophysical situations, the gravitational effects on light rays is weak, and the source and observer are far away from the lens, therefore they are studied under the formalism of weak and thin gravitational lenses. The basic and familiar variables in this discussion are shown in figure I.

In this framework the lens equation reads

$$\beta^a = \theta^a - \frac{d_{ls}}{d_s} \alpha^a. \quad (1)$$

The differential of this equation can be written as

$$\delta\beta^a = A_b^a \delta\theta^b, \quad (2)$$

where the matrix  $A_b^a$  is in turn expressed by

$$A_b^a = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}; \quad (3)$$

where the optical scalars  $\kappa$ ,  $\gamma_1$  and  $\gamma_2$ , are known as convergence  $\kappa$  and shear components  $\{\gamma_1, \gamma_2\}$ , and have the information of distortion of the image of the source due to the lens effects.

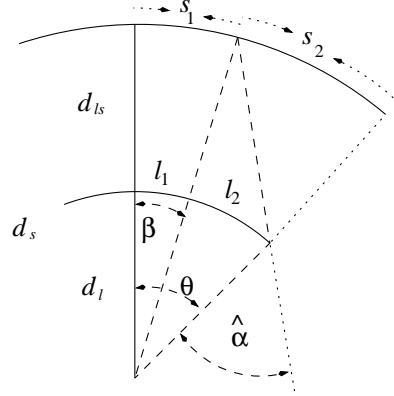


FIG. 1. This graph shows the basic and familiar angular variables in terms of a simple flat background geometry. The letter  $s$  denote sources, the letter  $l$  denotes lens and the observer is assumed to be situated at the apex of the rays.

It is somehow striking that in most astronomical works on weak gravitational lensing, it is assumed that the lens scalars and deflection angle, can be obtained from a Newtonian-like potential function. These expressions although are easy to use, have some limitations:

- They neglect more general distribution of energy-momentum tensor  $T_{ab}$ , in particular they only take into account the timelike component of this tensor. In this way they severely restrict the possible candidates to dark matter that can be studied with these expressions.
- They are not expressed in terms of gauge invariant quantities.
- Since these expressions are written in terms of a potential function, it is not easily seen how different components of  $T_{ab}$  contribute in the generation of these images.

Moreover, most of them assume from the beginning that thin lens is a good approximation.

In other cases in which the thin lens approximation is not used[BBV10], the results are presented in a way in which gauge invariance is not obvious.

Here we extend the work appearing in standard references on gravitational lensing[SEF92, SSE94, Wammb, Bar10] and present new expressions that do not suffer from the limitations mentioned above. In particular we present gauge invariant expressions for the optical scalars and deflection angle for some general class of matter distributions.

## II. INTEGRATED EXPANSION AND SHEAR

### A. General equations: The geodesic deviation equation

Let us consider the general case of a null geodesic starting from the position  $p_s$  (source) and ending at  $p_o$  (observer). Let us characterize the tangent vector as  $\ell = \frac{\partial}{\partial \lambda}$ ; so that

$$\ell^b \nabla_b \ell^a = 0; \quad (4)$$

that is,  $\lambda$  is an affine parameter.

We can now consider also a continuous set of nearby null geodesics. This congruence of null geodesics can be constructed in the following way. Let  $S$  be a two dimensional spacelike surface (the source image) such that the null vector  $\ell$  is orthogonal to  $S$ . Next we can generalize  $\ell$  to be a vector field in the vicinity of the initial geodesic in the following way: let the function  $u$  be defined so that it is constant along the congruence of null geodesics emanating orthogonally to  $S$  and reaching the observing point  $p_o$ . Then, without loss of generality we can assume that

$$\ell_a = \nabla_a u; \quad (5)$$

which implies that the congruence has zero twist.

We can complete to a set of null tetrad, so that  $m^a$  and  $\bar{m}^a$  are tangent to  $S$ . Then a deviation vector at the source image can be expressed by

$$\zeta^a = \varsigma \bar{m}^a + \bar{\varsigma} m^a. \quad (6)$$

In order to propagate this deviation vector along the null congruence one requires, that its Lie derivate vanishes along the congruence; that is

$$\mathfrak{L}_\ell \zeta^a = 0. \quad (7)$$

Using the GHP notation[GHP73] it can be shown that the previous equation can be written as

$$\mathbb{P}(\varsigma) + \zeta \rho + \bar{\varsigma} \sigma = 0; \quad (8)$$

where  $\mathbb{P}$  is the well behaved derivation of type  $\{1, 1\}$  in the direction of  $\ell$  (the null geodesic vector of the congruence).

Defining  $\mathcal{X}$  by

$$\mathcal{X} = \begin{pmatrix} \varsigma \\ \bar{\varsigma} \end{pmatrix}; \quad (9)$$

one can prove that  $\varsigma$  satisfy

$$\ell(\ell(\mathcal{X})) = -Q\mathcal{X}; \quad (10)$$

where  $Q$  is given by

$$Q = \begin{pmatrix} \Phi_{00} & \Psi_0 \\ \bar{\Psi}_0 & \Phi_{00} \end{pmatrix}; \quad (11)$$

with

$$\Phi_{00} = -\frac{1}{2} R_{ab} \ell^a \ell^b, \quad (12)$$

and

$$\Psi_0 = C_{abcd} \ell^a m^b \ell^c m^d. \quad (13)$$

Therefore, this form of the equation only involves curvature quantities.

Although the last equation can be integrated numerically without problems; it is sometimes convenient to have at hand some method for approximated solutions. So, next we present an approximation scheme that it can be applied to any order one wishes to obtain; although we will concentrate on the linear approximation since in weak lens studies it is consistent to consider linear effects of the curvature on geodesic deviations.

Let us first transform to a first order differential equation. Defining  $\mathcal{V}$  to be

$$\mathcal{V} \equiv \frac{d\mathcal{X}}{d\lambda}; \quad (14)$$

and

$$\mathbf{X} \equiv \begin{pmatrix} \mathcal{X} \\ \mathcal{V} \end{pmatrix}; \quad (15)$$

one obtains

$$\ell(\mathbf{X}) = \frac{d\mathbf{X}}{d\lambda} = \begin{pmatrix} \mathcal{V} \\ -Q\mathcal{X} \end{pmatrix} = A\mathbf{X}; \quad (16)$$

with

$$A \equiv \begin{pmatrix} 0 & \mathbb{I} \\ -Q & 0 \end{pmatrix}. \quad (17)$$

Equation (16) can be reexpressed in integral form, which gives

$$\mathbf{X}(\lambda) = \mathbf{X}_0 + \int_{\lambda_0}^{\lambda} A(\lambda') \mathbf{X}(\lambda') d\lambda'. \quad (18)$$

One can define the sequence

$$\mathbf{X}_1(\lambda) = \mathbf{X}_0 + \int_{\lambda_0}^{\lambda} A(\lambda') \mathbf{X}_0 d\lambda', \quad (19)$$

$$\mathbf{X}_2(\lambda) = \mathbf{X}_0 + \int_{\lambda_0}^{\lambda} A(\lambda') \mathbf{X}_1(\lambda') d\lambda'; \quad (20)$$

and so on.

Assuming that  $Q$  is in some sense small, one expects that this sequence will converge and therefore provide for the solution.

The complete linear iteration is given by

$$\mathbf{X}_3(\lambda) = \begin{pmatrix} \mathbb{I} - \int_{\lambda_0}^{\lambda} \int_{\lambda_0}^{\lambda'} Q'' d\lambda'' d\lambda' & (\lambda - \lambda_0)\mathbb{I} - \int_{\lambda_0}^{\lambda} \int_{\lambda_0}^{\lambda'} (\lambda'' - \lambda_0) Q'' d\lambda'' d\lambda' \\ - \int_{\lambda_0}^{\lambda} Q' d\lambda' & \mathbb{I} - \int_{\lambda_0}^{\lambda} (\lambda' - \lambda_0) Q' d\lambda' \end{pmatrix} \mathbf{X}_0; \quad (21)$$

Now in order to integrate the geodesic deviation equation, we must choose the correct initial conditions. In the case of light rays belonging to the past null cone of the observer and intersecting  $S$  at the source, this initial conditions are  $\mathcal{X} = 0$  and  $\mathcal{V} \neq 0$ ; since one can think the beam, starts backwards in time from the observer position, and so initially has vanishing departure, but with nonzero expansion and shear.

Therefore in the linear approximation one has

$$\mathcal{X}(\lambda) = ((\lambda - \lambda_0)\mathbb{I} - \int_{\lambda_0}^{\lambda} (\lambda - \lambda')(\lambda' - \lambda_0) Q' d\lambda') \mathcal{V}(\lambda_0); \quad (22)$$

and

$$\mathcal{V}(\lambda) = \left( \mathbb{I} - \int_{\lambda_0}^{\lambda} (\lambda' - \lambda_0) Q' d\lambda' \right) \mathcal{V}(\lambda_0). \quad (23)$$

In these integrations  $\lambda_0$  indicates the position at the observer and from now on,  $\lambda_s$  will indicate the position at the source.

We observe from the first expression, that if the metric were flat ( $Q = 0$ ), in order to get a deviation vector constructed from  $\mathcal{X}_1$ , defined as  $\mathcal{X}$  evaluated at  $\lambda_s = \lambda_0 + d_s$ , one must choose as initial condition

$$\mathcal{V}(\lambda_0) = \frac{1}{(\lambda_s - \lambda_0)} \mathcal{X}(\lambda_s = \lambda_0 + d_s) = \frac{1}{d_s} \mathcal{X}_1. \quad (24)$$

But in the case of the presence of a gravitational lens, if an observer sees an image of “size”  $\mathcal{X}_o$ , which means  $\mathcal{X}_o \equiv d_s \mathcal{V}_o$  (since actually what is observed is  $\mathcal{V}_o = \mathcal{V}(\lambda_0)$ ), then it should be produced by a source of size  $\mathcal{X}_s = \mathcal{X}(\lambda_s)$ , as described by equation (22).

We set from now on  $\lambda_0 = 0$  and  $\lambda_s = d_s$ , then eq.(22) reduces to

$$\mathcal{X}_s = \left( \mathbb{I} - \frac{1}{d_s} \int_0^{d_s} \lambda'(d_s - \lambda') Q' d\lambda' \right) \mathcal{X}_o. \quad (25)$$

## B. Optical scalars in terms of curvature

In order to compare with the standard representation of the lens scalar we note that the original deviation vector in the source will be given by eq.(25), i.e.

$$\begin{pmatrix} \varsigma_s \\ \bar{\varsigma}_s \end{pmatrix} = \left( \mathbb{I} - \int_0^{d_s} \frac{\lambda'(d_s - \lambda')}{d_s} Q' d\lambda' \right) \begin{pmatrix} \varsigma_o \\ \bar{\varsigma}_o \end{pmatrix}; \quad (26)$$

if we make the following decomposition into real and imaginary part,

$$\varsigma_o = \varsigma_{oR} + i\varsigma_{oI}, \quad (27)$$

$$\varsigma_s = \varsigma_{sR} + i\varsigma_{sI}, \quad (28)$$

$$\Psi_0 = \Psi_{0R} + i\Psi_{0I}; \quad (29)$$

we obtain from eq.(26) that

$$\begin{aligned} \varsigma_{sR} &= \left( 1 - \int_0^{d_s} \frac{\lambda'(d_s - \lambda')}{d_s} (\Phi'_{00} + \Psi'_{0R}) d\lambda' \right) \varsigma_{oR} - \left( \int_0^{d_s} \frac{\lambda'(d_s - \lambda')}{d_s} \Psi'_{0I} d\lambda' \right) \varsigma_{oI}, \\ \varsigma_{sI} &= \left( 1 - \int_0^{d_s} \frac{\lambda'(d_s - \lambda')}{d_s} (\Phi'_{00} - \Psi'_{0R}) d\lambda' \right) \varsigma_{oI} - \int_0^{d_s} \frac{\lambda'(d_s - \lambda')}{d_s} \Psi'_{0I} d\lambda' \varsigma_{oR}. \end{aligned} \quad (30)$$

Note also that in principle the integration must be made through the actual geodesic followed by a photon in its path from the source to observer. However the last expressions are valid only in the limit where the linear approximation is valid. If one considers a linear perturbation from flat spacetime, then the curvature components  $\Phi_{00}$  and  $\Psi_0$  would be already of linear order. Then, in the context of weak gravitational lensing, it is consistent to consider a null geodesic in flat spacetime; since the actual null geodesic can be thought as a null geodesic in flat spacetime plus some corrections of higher orders.

Now, in order to compare with the usual expressions for the lens scalars  $\kappa, \gamma_1$  and  $\gamma_2$ , let us recall that they are defined via the relation eq.(2); but since it is a linear relation, one can relate the deviation vectors by the same matrix, namely

$$\varsigma_s^i = A_{ij}^i \varsigma_o^j; \quad (31)$$

where  $\{\varsigma_s^i, \varsigma_o^i\}$  are the spatial vector associated with  $\{\varsigma_s, \varsigma_o\}$  respectively.

Therefore, by replacing into eq.(31), we obtain

$$\varsigma_{sR} = (1 - \kappa - \gamma_1) \varsigma_{oR} - \gamma_2 \varsigma_{oI}, \quad (32)$$

$$\varsigma_{sI} = -\gamma_2 \varsigma_{oR} + (1 - \kappa + \gamma_1) \varsigma_{oI}; \quad (33)$$

which by comparing with eq.(30), implies that

$$\kappa = \frac{1}{d_s} \int_0^{d_s} \lambda' (d_s - \lambda') \Phi'_{00} d\lambda', \quad (34)$$

$$\gamma_1 = \frac{1}{d_s} \int_0^{d_s} \lambda' (d_s - \lambda') \Psi'_{0R} d\lambda', \quad (35)$$

$$\gamma_2 = \frac{1}{d_s} \int_0^{d_s} \lambda' (d_s - \lambda') \Psi'_{0I} d\lambda'. \quad (36)$$

Let us emphasize that these expressions for the weak lens quantities are explicitly gauge invariant, since they are given in terms of the curvature components. This is in contrast to the usual treatment of weak lensing found in the literature, which use for example equation (2.17) of reference [SEF92] as the source for the calculation of the lens scalars.

Note that these expressions are valid for any weak field gravitational lens on a perturbed at spacetime, without restriction on the size of the lens compared with the other distances. As a final comment to this section it is important to remark that these expressions are valid for any weak gravitational lensing on a perturbed flat spacetime, without restriction on the size of the lens compared with the other distances. However, if we make use of the hypothesis of thin lens, these equations can be further simplified, as we will show below.

### III. THE THIN LENS APPROXIMATION

#### A. The general case

Now, we will consider the case of a lens whose size is small compared with the distances to the source and the observer. Let us consider a null geodesic coming from a source located at a distance  $d_s$  from the observer, and at a distance  $d_{ls}$  from the lens, coming parallel to the  $y$  axis, but in the negative direction; we will use  $J$  to represent the impact parameter and  $\vartheta$  to denote the angle of the trajectory as measured from the  $z$  axis, in the  $(z, x)$  plane.

Then if we assume a thin lens,  $\Phi_{00}$  and  $\Psi_0$  will be sharply peaked around  $\lambda = d_l$ , where it is located and the expressions for the lens scalars are reduced to

$$\kappa = \frac{d_l d_{ls}}{d_s} \hat{\Phi}_{00}, \quad (37)$$

$$\gamma_1 + i\gamma_2 = \frac{d_l d_{ls}}{d_s} \hat{\Psi}_0, \quad (38)$$

where

$$\begin{aligned} \hat{\Phi}_{00} &= \int_0^{d_s} \Phi_{00} d\lambda, \\ \hat{\Psi}_0 &= \int_0^{d_s} \Psi_0 d\lambda, \end{aligned} \quad (39)$$

are the projected curvature scalars along the line of sight.

We again emphasize that these expressions for the weak lens scalars are explicitly gauge invariants.

#### B. The axially symmetric case

##### 1. The lens scalars in terms of projected Ricci and Weyl Scalars

For axially symmetric lens, the projected curvature scalars are given by

$$\hat{\Phi}_{00}(J) = \int_0^{d_s} \Phi_{00}(\lambda') d\lambda', \quad (40)$$

$$\hat{\Psi}_0(J) = -e^{2i\vartheta} \hat{\psi}_0(J). \quad (41)$$

where one can see that

$$\hat{\psi}_0(J) = -e^{-2i\vartheta} \int_0^{d_s} \Psi_0(\lambda') d\lambda'. \quad (42)$$

The reason for the minus sign choice is that in many common astrophysical situations one would find  $\hat{\psi}_0(J) > 0$ .

By replacing in eqs. (37) and (38), we obtain for the lens scalars

$$\kappa = \frac{d_{ls} d_l}{d_s} \hat{\Phi}_{00}(J), \quad (43)$$

$$\gamma_1 = -\frac{d_{ls} d_l}{d_s} \hat{\psi}_0(J) \cos(2\vartheta), \quad (44)$$

$$\gamma_2 = -\frac{d_{ls} d_l}{d_s} \hat{\psi}_0(J) \sin(2\vartheta), \quad (45)$$

which implies that

$$\gamma = \frac{d_{ls} d_l}{d_s} \hat{\psi}_0(J). \quad (46)$$

These equations can be compared to those of reference [FKN00]; where they use different notation but similar content.

##### 2. Deflection angle in terms of projected Ricci and Weyl Scalars

We wish now to express the deflection angle in terms of the curvature scalars.

If we define the components of  $\alpha^i = (\alpha^1, \alpha^2)$  as

$$(\alpha^i) = \alpha(J) \left( \frac{z_0}{J}, \frac{x_0}{J} \right); \quad (47)$$

then, it can be shown that the optical scalars can be written in terms of  $\alpha(J)$  as

$$\kappa = \frac{1}{2} \frac{d_{ls} d_l}{d_s} \left( \frac{d\alpha}{dJ} + \frac{\alpha(J)}{J} \right), \quad (48)$$

$$\gamma_1 = \frac{1}{2} \frac{d_{ls} d_l}{d_s} \cos(2\vartheta) \left( \frac{d\alpha}{dJ} - \frac{\alpha(J)}{J} \right), \quad (49)$$

$$\gamma_2 = \frac{1}{2} \frac{d_{ls} d_l}{d_s} \sin(2\vartheta) \left( \frac{d\alpha}{dJ} - \frac{\alpha(J)}{J} \right). \quad (50)$$

It is interesting to note that

$$\kappa - \gamma_1 \cos(2\vartheta) - \gamma_2 \sin(2\vartheta) = \frac{d_l d_{ls}}{d_s} \frac{\alpha(J)}{J}; \quad (51)$$

from which, using eqs. (43-45), it is deduced that

$$\alpha(J) = J(\hat{\Phi}_{00}(J) + \hat{\psi}_0(J)). \quad (52)$$

It is worthwhile to remark that this constitutes an equation for the bending angle expressed in terms of the gauge invariant curvature components in a very simple compact form.

#### IV. DETAILED STUDY OF STATIONARY SPHERICALLY SYMMETRIC LENSES

A stationary spherically symmetric spacetime can be expressed in terms of the standard line element

$$ds^2 = a(r)dt^2 - b(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (53)$$

where it is convenient to define  $\Phi(r)$  and  $m(r)$  from

$$a(r) = e^{2\Phi(r)}, \quad (54)$$

and

$$b(r) = \frac{1}{1 - \frac{2m(r)}{r}}. \quad (55)$$

For our purpose, it is more convenient to use a null coordinate system to describe the spherically symmetric geometry. Let us introduce then, a function

$$u = t - r^*; \quad (56)$$

where  $r^*$  is chosen so that  $u$  is null. Then by inspection of equation (53) one can see that

$$du = dt - \frac{dr^*}{dr} dr = dt - \sqrt{\frac{b}{a}} dr; \quad (57)$$

since then one has

$$ds^2 = a du^2 + 2\sqrt{ab} du dr - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (58)$$

It is natural to define the principal null direction  $\tilde{\ell}_P$  from

$$\tilde{\ell}_P = du; \quad (59)$$

which implies that the vector is

$$\tilde{\ell}_P^a = g^{ab} du_b = \frac{1}{\sqrt{ab}} \left( \frac{\partial}{\partial r} \right)^a. \quad (60)$$

We complete to a null tetrad with

$$\tilde{n}_P = \frac{\partial}{\partial u} + U \mathbb{A} \frac{\partial}{\partial r}, \quad (61)$$

with the complex null vector

$$\tilde{m}_P = \frac{\sqrt{2}P_0}{r} \frac{\partial}{\partial \zeta}; \quad (62)$$

in terms of the stereographic coordinate  $\zeta$ .

Therefore, one has

$$\mathbb{A} = \frac{1}{\sqrt{ab}}, \quad (63)$$

and

$$U = -\frac{1}{2b\mathbb{A}^2} = -\frac{a}{2}. \quad (64)$$

The more general distribution of energy-momentum compatible with spherical symmetry is given by

$$T_{tt} = \varrho e^{2\Phi(r)}; \quad (65)$$

$$T_{rr} = \frac{P_r}{\left(1 - \frac{2m(r)}{r}\right)}; \quad (66)$$

$$T_{\theta\theta} = P_t r^2; \quad (67)$$

$$T_{\varphi\varphi} = P_t r^2 \sin^2(\theta); \quad (68)$$

where we have introduced the notion of radial component  $P_r$  and tangential component  $P_t$ .

The Einstein field equations

$$G_{ab} = -8\pi T_{ab}, \quad (69)$$

in terms of the previous variables are

$$\frac{dM}{dr} = 4\pi r^2 \varrho, \quad (70)$$

$$r^2 \frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P_r}{1 - \frac{2M(r)}{r}}, \quad (71)$$

$$\begin{aligned} & r^3 \left( \frac{d^2\Phi}{dr^2} + \left( \frac{d\Phi}{dr} \right)^2 \right) \left( 1 - \frac{2M}{r} \right) \\ & + r^2 \frac{d\Phi}{dr} \left( 1 - \frac{M}{r} - \frac{dM}{dr} \right) \\ & - r \frac{dM}{dr} + M = 8\pi r^3 P_t. \end{aligned} \quad (72)$$

One can show that the Ricci scalars defined with respect to the tetrad  $\{\tilde{\ell}_P, \tilde{n}_P, \tilde{m}_P, \tilde{\bar{m}}_P\}$  are given by

$$\tilde{\Phi}_{00} = \frac{4\pi}{a} (\varrho + P_r), \quad (73)$$

$$\tilde{\Phi}_{11} = \pi (\varrho - P_r + 2P_t), \quad (74)$$

$$\tilde{\Phi}_{22} = a\pi (\varrho + P_r), \quad (75)$$

$$\tilde{\Lambda} = \frac{\pi}{3} (\varrho - P_r - 2P_t); \quad (76)$$

and the Weyl scalar,

$$\tilde{\Psi}_2 = \frac{4\pi}{3}(\varrho - P_r + P_t) - \frac{m}{r^3}. \quad (77)$$

Now, when one made a tetrad transformation between the spherically symmetric tetrad and the tetrad adapted to the photon geodesic, one find that the function  $\alpha(J)$  expressed in terms of the spherically symmetric null tetrad reads,

$$\alpha(J) = J \int_{-d_l}^{d_{ls}} \left[ -\frac{3J^2}{r^2} \tilde{\Psi}_2 + \frac{2J^2}{r^2} (\tilde{\Phi}_{11} - \frac{1}{4} \tilde{\Phi}_{00}) + \tilde{\Phi}_{00} \right] dy. \quad (78)$$

Note that in this case, the integration is on the coordinate  $y$ , instead of using arbitrary affine parameter. Also note that  $r = \sqrt{J^2 + y^2}$ .

This constitutes an important explicit relation for the bending angle in terms of the curvature as seen in an spherically symmetric frame; which is the natural frame for the sources of the gravitational lens.

In terms of the physical energy-momentum tensor it is obtained

$$\alpha(J) = J \int_{-d_l}^{d_{ls}} \left[ \frac{3J^2}{r^2} \left( \frac{m(r)}{r^3} - \frac{4\pi}{3} \varrho(r) \right) + 4\pi (\varrho(r) + P_r(r)) \right] dy. \quad (79)$$

In a similar way, the lens scalars, in terms of the

spherically symmetric physical fields, are given by

$$\begin{aligned} \kappa &= \frac{4\pi d_l d_{ls}}{d_s} \int_{-d_l}^{d_{ls}} \left[ \rho + P_r + \frac{J^2}{r^2} (P_t - P_r) \right] dy \\ \gamma &= \frac{d_l d_{ls}}{d_s} \int_{-d_l}^{d_{ls}} \frac{J^2}{r^2} \left[ 4\pi (\rho + P_t - P_r) - \frac{3m}{r^3} \right] dy. \end{aligned} \quad (80)$$

These new expressions let us see explicitly the contributions of different components of the energy-momentum tensor on the optical scalars. Let us note that a couple of terms disappear in the isotropic case in which  $P_r = P_t$ .

## V. FINAL COMMENTS

Several works on weak gravitational lensing reach up to the expressions that relate the optical scalars with the curvature components in terms of the tetrad adapted to the motion of the photons; we have here also presented expressions for the bending angle in terms of the curvature components. Furthermore, we have presented above expressions for the optical scalars and deflection angle directly in terms of the matter components of the sources of the gravitational lens, valid for an extended class of matter distributions. Our expressions circumvent several deficiencies as are: gauge dependence, lack of explicit expressions, neglect of spacelike components of the energy-momentum tensor, etc. The extension of this study to sources with different structure and to the cosmological background will be presented elsewhere.

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